ON SOME ASPECTS OF FUNCTIONAL REPRODUCIBILITY OF MULTIVARIABLE LINEAR DYNAMICAL SYSTEMS

A THESIS

Presented to

The Faculty of the Division of Graduate

Studies and Research

By

Kuo-ting Tao

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Operations Research

Georgia Institute of Technology

July, 1973

ON SOME ASPECTS OF FUNCTIONAL REPRODUCIBILITY

OF MULTIVARIABLE LINEAR DYNAMICAL SYSTEMS

Approved:

P. Zunde, Chairman

T. O. Kvålseth

M. S. Bazaraa

Date approved by Chairman: July 5, 73

ACKNOWLEDGMENTS

The author would like to express his sincere gratitude to Dr. Pranas Zude, who served as thesis advisor, for his helpful guidance and encouragement.

The author is also indebted to the members of the reading committee, Dr. M. S. Bazaraa and Dr. T. O. Kvalseth, for their valuable comments and suggestions. The excellent typing by Mrs. Jackie Van Hook is also thanked here.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

SUMMARY

In this work, some basic aspects of functional reproducibility and input functional observability of multi-variable linear dynamical systems such as dependencies on homogeneous solution and on time-intervals, structures of function space, admissibility of input and output functions, functional attainability, dual relations, etc., are investigated. Attentions are concentrated on linear time-invariant dynamical systems, and problems are handled and interpreted from the frequency domain point of view.

The theoretical foundations are based on the formalization of a finite-dimensional rational function vector space over a rational function field. This formalization which was not used by other authors in this context, gives a powerful tool to investigate the properties of functional reproducibility and input functional observability. In particular, results have been obtained regarding the effects of feed-back compensation, and existence and characterization of functionally reproducible and input functionally observable systems. Besides, a design proposal for a pseudo-inverse system without differentiators, which is proved to be able to approximate the true inverse system to any desired precision, is presented.

CHAPTER I

INTRODUCTION

The objects of investigation of this thesis are mathematical models of physical or real life systems whose behavior may be adequately described by the type of mathematical structure called dynamical systems, which shall be denoted by Σ, in general. Roughly speaking, a system can be viewed as "a structure into which something (matter, energy, or information) may be put at certain times and which itself put out something at certain times" [1]. Hence, the concept of a system Σ includes an associated time set T and at each moment of time t in T, the system receives some input $\underline{u}(t)$ and emits some output $\underline{y}(t)$. In general, the output of the system Σ depends on both the present input and the past history of Σ, i.e. the state $x(t)$ of Σ. Thus the state of Σ can be intuitively defined as that part of the present and past history of Σ which is relevant to the determination of present and future outputs [1]. A precise technical definition of a dynamical system is given in IV-1.

Standard problems of dynamical systems theory are problems such as, stability, control, state reconstruction, optimization, equivalence, decomposition and synthesis. This thesis addresses itself to the fundamental questions of existence, properties and construction of dynamical systems which are functional reproducible. Roughly speaking, a dynamical system is functionally reproducible if it has the capability to generate specified time function [2]. In many industrial situations the aim of control is to make the output vector of a plant take a certain form as a function of time [3].

A corresponding problem which is also studied in this work may be stated as the capability of uniquely determining the input function over a time interval by using the measured output function. This is known as invertibility in the literature and is of interest in many cases of applications. One application is for information recovery in coding theory [4,5,6], that is we want to unambiguously recover the original information from the information received which has passed through an encoder for the purpose of transmission.

Another application is in decoupling in multi-variable systems [6,7]. In the design of multi-variable systems, it is often of interest to know whether or not it is possible to have a single input influence a single output. Falb and Wolovich [7] have used a feedback technique to decouple a multi-variable system. It has been shown that the necessary and sufficient condition for decoupling is also a sufficient condition for functional reproducibility.

Invertibility finds application in the area of filtering and estimation in the presence of colored noise. Anderson and Moore [8] have determined a whitening filter which is a dynamical inverse of a dynamical model for a random process, and have used this tool to simplify the derivation of Kalman filter. Actually, one can see the potential of application of inverse system to adaptive filtering and control, and many other diagnostic techniques in engineering and life system.

Besides, Sain [9] has shown that the Cruz-Perkins sensitivity matrix derivation [10] depends upon an assumption of invertibility.

CHAPTER II

LITERATURE SURVEY

The problem of functional reproducibility of multi-variable dynamical system was first investigated by Brockett and Mesarovic [2], who gave the definition and a necessary and sufficient condition for the functional reproducibility in terms of the linear time-invariant system parameters.

The derivation of the necessary and sufficient condition came from the understanding that a given linear time-invariant system is functionally reproducible if and only if the row vectors of the transfer function matrix of the system are linearly independent. Now, since the entries of the transfer function matrix are rational functions of complex variable s, it may be referred that independence was meant in the context of a vector space over the field of rational functions, and, in particular, the vector space so defined is finite-dimensional.

Although this concept was used implicitly by Brockett and Mesarovic, no explicit formalization was made. In this work, however, the use of the finite-dimensional rational function vector space over rational field is justified, and then various problems will be considered from this point of view.

A more restrictive version of functional reproducibility concerning the generation of the class of r-times differentiable functions by the class of (r-2) - times differentiable input functions, has been discussed by Birta and Mufti [11] in 1966. As expected, this problem is harder than the usual functional reproducibility problem and it seems not so easy to get a more generalized result, such as, generate r-times differentiable functions by (r-k) - times differentiable functions, where $2 < k < r$.

From an application oriented point of view, the strongly related question of inverting a linear dynamical system has been of interest to the control engineer for many years. This is understandable because one can see that in many cases, people would like to find out the <u>causes</u> (inputs) which induce the <u>effects</u> (outputs).

The earliest example of the application of this inverse concept is the work by Bode and Shannon [12] in 1950, who used a non-state variable whitening filter to simplify the derivation of classical Wiener filter.

Since the <u>inverse system</u> or the <u>invertibility</u> problem is concerned with the exact determination of the unknown input by using the knowledge the output function, after some consideration, one can see that the invertibility problem is related to the functional reproducibility problem in a dual sense. Therefore, there is no doubt that Brockett and Mesarovic did make a great contribution to this old problem. Besides, Brockett [13] discussed an algorithm for constructing the inverse system of a single-input single-output system in 1965 which is actually a method to find out the transfer function without solving the characteristic equation.

It seems that the first general approach to the problem of inverse system was due to Youla and Dorato [14] in 1966. They proposed a method to test invertibility and developed an inversion algorithm, based on the manipulation of the system coefficient matrices. This criterion is simpler than Brockett and Mesarovic's and this is an important paper on which some other works were based. Besides, Silverman [15,16] developed a sequential algorithm which can test the invertibility and construct an inverse system at the same time.

Another approach was due to Sain and Mussey [17]. They studied the invertibility of linear sequential circuit first, and introduced the concept of L-delay invertibility, i.e. if the unknown input segment

$$\hat{U}_K = \{\underline{u}_0, \underline{u}_1, \dots \underline{u}_K\}$$

can be recognized exactly by using the output segment

$$\hat{Y}_{K+L} = \{\underline{y}_0, \underline{y}_1, \dots \underline{y}_K, \underline{y}_{K+1}, \dots \underline{y}_{K+L}\}$$

where L is a positive integer less than the dimension of the state space.

The discrete-time character of the linear sequential circuit makes the problem easier to handle, so that simpler test and algorithm were developed. Sain and Massey also studied continuous-time systems, using the concept <u>Associated Sequential Circuit</u> [18]. Corresponding to the L-delay invertibility, they introduced the L-integral invertibility for the continuous-time case, i.e. the capability of exactly recognizing the L-th integral of the unknwon input by using the information of the output. A theorem stating that the given continuous-time system is invertible if and only if the associated sequential circuit is, has been proved and an algorithm for inverse system construction has been extended to the continuous-time case.

So far, the systems have been talked about are linear time-invariant systems, and it seems that, up to now, most of the invertibility and inverse construction problems for this kind of problems have been solved.

The problem becomes more difficult for linear time-varying system. However, some special cases have been studied. In 1968, Silverman [15] got some result for the single-input single-output case, with some restriction, which was an extension of Brockett's algorithm [13] for the time-invariant system. Next year, Silverman [16] obtained a result for multi-variable system, but with more restrictions.

As to non-linear systems, some results appeared in Brockett and Mesarovic's original paper. Approached from the aspect of functional reproducibility, as expected, that was for almost linear systems only.

Generally speaking, it is not easy to handle the functional reproducibility or invertibility problems for the case of time-varying or non-linear systems, since we do not have adequate knowledge of general integral operators.

Although probably the original motivation of the study of functional reproducibility might not be definite and clear, there is no doubt that it has influenced many other fields. Invertibility is the direct beneficiary and again it finds applications in the fields of filtering, coding, decoupling and sensitivity problems which have been discussed in Chapter I.

As a summary to the previous works, the following comments and remarks are made:

* All the previous studies were concerned with the functional reproducibility or invertibility, but the situation of functional non-reproducibility or non-invertibility were not discussed.

* The previous researchers were trying to find out criteria for functional reproducibility or invertibility, and algorithms for inverse systems construction, based on the manipulation of certain parameter matrices of the system.

* The basic idea, however, which comes from the operational transformation between the input function space and output function space was neither formalized nor emphasized. Although Brockett and Mesarovic [2] used this concept implicitly. Hence, the structural properties such as dimensionality, inner product, adjoint operator, decomposition, etc., were not discussed.

* The use of the terminology, <u>inverse system</u>, was not unified. Singh and Liu [6] classified the term of inverse system used by previous authors into post-inverse systems and pre-inverse systems, which will be explained in detail in Chapter VI.

To clarify these non-unified terminologies, throughout this work, the <u>inverse system</u> is used for post-inverse system and the name, <u>function generator</u>, is used for pre-inverse system.

* The dual concept for functional reproducibility and invertibility was used implicitly, without justification from the consideration of

inner product space, and dual operator, etc. Once again, since the function space structures were not discussed, a complete dual theorem was neither given nor interpreted in depth.

* From practical point of view, the major deficiency of the inverse system is that in most cases, ideal differentiators are not avoidable. Since there is no ideal differentiator, this prevents the applications of inverse system.

CHAPTER III

OBJECTIVES AND METHODS OF RESEARCH

In this work, only linear time-invariant dynamical systems will be considered. Problems will be approached primarily from frequency domain point of view. Frequency domain functions and transfer function matrices will be used to study the transformation between the input function space and output function space.

First of all, the set of all vector-valued rational functions will be formalized as a finite dimensional vector space over the field of rational functions. Then, by the results from finite-dimensional vector space, the argument of the existence of a right inverse of the transfer function matrix will deduce to a frequency domain criterion for functional reproducibility. Then, the use of a generalized inner product and adjoint operator, etc., will result in another frequency domain criterion which is a new result of this study. This kind of approach makes the problem look easier and clearer, and since many practical design works are approached from the frequency domain, this criterion is particularly suitable for hand calculation provided that the system size is not too large.

Then, a major class of functions, namely the rational functions in s, will be considered as input and output functions of interest and the structure of the finite-dimensional rational vector space will be used as a bridge to study the structural properties of the output function space. Then the concept of admissible input and output functions will be introduced and the functional-attainable set will be defined as the set of all admissible rational functions embedded in the rational function range subspace.

Within the above outlined framework, the following topics are dealt with in detail:

1. The dependencies of functional reproducibility on homogeneous solution and on time-interval will be clarified.
2. Procedures to find out the rank and basis of the transfer function matrix are discussed in detail.
3. Concept of functional attainability is introduced and non-reproducibility is emphasized. Structures of output function space are discussed.
4. Procedures to test if a given trajectory is attainable is given.
5. The so-called <u>invertibility</u> is re-defined in the time-domain, which makes it more precise. This is referred to as <u>input functional observability</u> in this work. Concept of functional-non-deducibility is introduced, and then a complete dual theorem is proved.
6. Inverse system construction is approached from this unified frequency domain view point.
7. The effects of feedback on functional reproducibility and input functional observability are discussed.
8. The condition for the existence of an input matrix (output matrix) such that the system can be functionally reproducible (invertible) is discussed.

Finally, since it has been shown that for systems which do not have direct input-output link in their canonical form, the inverse system must contain a certain number of differentiators which are inaccurate in practice such that the applicability of the inverse system is limited, the use of a dynamical inverse system (without differentiators) is suggested, in this work, to approximate the true inverse system (with differentiators). This approximating inverse system will be called dynamical pseudo-inverse system. A theorem concerning the convergence of this approximation is proved.

CHAPTER IV

SYSTEM THEORETICAL PRELIMINARIES

IV-1. Dynamical Systems

A finite-dimensional continuous-time dynamical system Σ is a mathematical object in the form of a set of differential equations

$$\begin{cases} \dot{\underline{x}}(t) = \underline{f}(\underline{x}, \underline{u}, t) \\ \underline{y}(t) = \underline{g}(\underline{x}, \underline{u}, t) \end{cases}$$

where $\underline{x}$ is a $p \times 1$ state vector

$\underline{y}$ is an $n \times 1$ output vector

$\underline{u}$ is an $m \times 1$ output vector

$\underline{f}$ is a $p \times 1$ vector-valued function

$\underline{g}$ is an $n \times 1$ vector-valued function

$\underline{t}$ is a real variable

and we assume $\underline{f}$ is smooth enough (e.g. satisfying Lipschitz condition [19]) such that the differential equation has a solution, if initial condition is given.

Similarly, a finite-dimensional discrete-time dynamical system Σ_d is given by a set of difference equations:

$$\begin{cases} \underline{x}_{k+1} = \underline{f}(\underline{x}_k, \underline{u}_k, k) \\ \underline{y}_k = \underline{g}(\underline{x}_k, \underline{u}_k, k) \end{cases}$$

where $\underline{x}_k$ = value of the p $\times$ 1 state vector $\underline{x}$ at time instant k

$\underline{y}_k$ = value of the n $\times$ 1 output vector $\underline{y}$ at time instant k

$\underline{u}_k$ = value of the m $\times$ 1 input vector $\underline{u}$ at time instant k

$\underline{f}$ = p $\times$ 1 vector-valued function

$\underline{g}$ = n $\times$ 1 vector-valued function

k = integers

The systems are called finite-dimensional, since at each instant, the dimensions of the states are finite. We call p the dimension of the state space. If $\underline{f}$ and $\underline{g}$ do not depend on t or k explicitly, we call the systems <u>time-invariant</u> or <u>autonomous</u>.

In this study, we shall be concerned only with the class of dynamical systems which can be modeled or approximated by linear differential or difference equations with constant coefficients.

A finite-dimensional continuous-time-invariant linear dynamical system S, is given by

$$\begin{cases} \dot{\underline{x}}(t) = \underline{A}\ \underline{x}(t) + \underline{B}\ \underline{u}(t) \\ \underline{y}(t) = \underline{C}\ \underline{x}(t) + \underline{D}\ \underline{u}(t) \end{cases}$$

where $\underline{x}$, $\underline{y}$, $\underline{u}$ are as before.

$\underline{A}$ is a p $\times$ p constant matrix.

$\underline{B}$ is a p $\times$ m constant matrix.

$\underline{C}$ is an n $\times$ p constant matrix.

$\underline{D}$ is an n $\times$ m constant matrix.

In the sequel, we will frequently use [$\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$] to denote the system S, and throughout this work, the dimensions of the matrices $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$ are unchanged.

For the time-invariant case, we can apply Laplace transform to the system equation, and, assuming zero initial condition, get

$$\underline{y}(s) = [\underline{C}(IS-\underline{A})^{-1} \underline{B} + \underline{D}]\underline{u}(s)$$

$$\triangleq \underline{H}(s)\ \underline{u}(s)$$

where $\underline{H}(s)$ is called the <u>transfer function matrix</u> of the system S. For finite-dimensional discrete-time-invariant linear dynamical system, S_d, the corresponding mathematical object is

$$\begin{cases} \underline{x}_{k+1} = \underline{A}\ \underline{x}_k + \underline{B}\ \underline{u}_k \\ \underline{y}_k \quad\ = \underline{C}\ \underline{x}_k + \underline{D}\ \underline{u}_k \end{cases}$$

The latter system is occasionally referred to as <u>time-invariant linear sequential circuit</u>. Given a continuous-time linear system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$, the corresponding discrete-time system $S_d = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is called the <u>associated linear sequential circuit</u> [18].

IV-2. <u>Controllability, Observability, Functional Reproducibility and Invertibility</u>

When studying or designing a dynamical system, it is important to be able to characterize the given system by referring to the existence of certain limitations of the system.

Among those properties, controllability and observability are the well-known concepts in the theory and practice of linear dynamical systems. Since, in this work, only the linear time-invariant systems are concerned, the following definitions are for time-invariant case.

Definition 1.

Given a linear time-invariant system $S = [\underline{A}\ \underline{B}\ \underline{C}\ \underline{D}]$ and initial state $\underline{x}_o$, the system S is said to be state controllable if and only if any state $\underline{x}_1$ can be reached in a finite time, by a control $\underline{u}(t)$.

If in the above definition, $\underline{x}_o$ and $\underline{x}_1$ are replaced by $\underline{y}_o$ and $\underline{y}_1$, then we are referring to output controllability.

Definition 2.

A linear time-invariant system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is said to be observable, if we can uniquely determine the initial state $\underline{x}_o$ of the system by observing the output $\underline{y}(t)$ and the input $\underline{u}(t)$ over a finite time interval.

It has been shown that if a system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is controllable, then a corresponding system $S^* = [-\underline{A}', -\underline{C}', \underline{B}', \underline{D}']$ is observable, and vice versa. This is known as Dual Theorem.

A much stronger property than that of controllability is the capability of reproducing any desired reasonable output trajectory over some time interval, using unconstrained control input. This is known as functional reproducibility, first introduced by Brockett and Mesarovic [2] in 1964.

Unlike the controllability problem which is concerned with the onto-ness of a linear transformation between two real Euclidean spaces, the functional reproducibility is concerned with the onto-ness of a linear operator which transforms one function space (input function space) to another (output function space).

On the other hand, a corresponding concept may be stated as the capability of exactly determining the input function over a time interval, by the given initial state and measured output function. While this property is referred to as invertibility in the literature, it will be re-defined as input functional observability in this work, and it will be shown that dual theorem exists between functional reproducibility and input functional observability, which is very analogous to that for controllability and observability.

Rigorous definitions for functional reproducibility and input functional observability will appear in V - 1 and VI respectively.

CHAPTER V

FUNCTIONAL REPRODUCIBILITY AND FUNCTIONAL ATTAINABLE SET

V-1. General Definitions and Properties

Consider the time-invariant dynamical system Σ:

$$\underline{\dot{x}}(t) = \underline{f}(\underline{x}, \underline{u})$$

$$\underline{y}(t) = \underline{g}(\underline{x}, \underline{u})$$

and denote the output solution subject to initial condition $\underline{x}(0) = \underline{x}_o$ by $\varphi(x_o, \underline{u}, t)$.

The following norms are defined as measurement standards:

If $\underline{A}$ is a constant matrix, then $|\underline{A}| \triangleq \sum_i \sum_j |a_{ij}|$, where a_{ij} are elements of $\underline{A}$, is a valid matrix norm [20].

If $\underline{x}$ is a vector-valued function defined on $[o, \infty)$, then $||\underline{x}|| \triangleq \sup_{t\epsilon[o,\infty)} |\underline{x}(t)|$ is a valid norm.

If $\underline{x}$ is a k-times differentiable vector-valued function on $[o,\infty)$, $||\underline{x}||_k = \max_{o\leq i\leq k} ||\underline{x}^{(i)}||$ is a norm, where

$\underline{x}^{(i)}$ is the i-th derivative of $\underline{x}(t)$.

The following definition is after Brockett and Mesarovic [2].

Definition 1

A homogeneous solution $\varphi(\underline{x}_o, \underline{o}, t)$ of the system Σ is said to be functionally reproducible (F.R.) if and only if for any $\eta > 0$ and any $\tau > 0$, there exists a $\delta(\eta,\tau) > 0$ such that for every $\underline{y}$

$$\|\underline{y}(t) - \varphi(\underline{x}_o, \underline{0}, t)\|_p < \delta(\eta, \tau)$$

where p is the dimension of the state space, there exists a function $\underline{u}$, $\|\underline{u}\| \le \eta$, such that $\varphi(\underline{x}_o, \underline{u}, t) = \underline{y}(t)$ for every t in $[0,\tau]$.

In the sequel, we will use F.R. instead of functionally reproducible or functional reproducibility. Intuitively speaking, the F.R. of a given homogeneous solution $\varphi(\underline{x}_o, \underline{0}, t)$ means that there always exists a <u>neighborhood</u> of $\varphi(\underline{x}_o, \underline{0}, t)$ which may depend on how much control effort we want to use (i.e. η), and how long a time we want to control the output (i.e. τ), such that any output trajectory in the neighborhood can be reached by some bounded control function $\underline{u}$. The concept of F.R. is a local concept, since only the behavior of the system in a small neighborhood of a known solution is discussed.

The purpose of using this kind of norm, $\|\cdot\|_p$, is to exclude the possibility of involving impulse input functions. For example, for a single-input, single-output linear system with transfer function $\frac{1}{s^2 + 3s + 2}$ (p = 2), the trajectories of the forms, $\frac{1}{s^2 + as + b}$ (one-time differentiable) and $\frac{1}{s + c}$ (not differentiable at t = 0), cannot be generated by any bounded function. So, it is meaningless for F.R., if the restriction of differentiability were not made.

It is not true, in general, that F.R. means any desired trajectory with appropriate differentiability can be reached from a given homogeneous solution, although it will be shown to be true for linear dynamical systems.

<u>Definition 2</u>

A system Σ is said to be a F.R. system if and only if all of its homogeneous solutions are F.R.

Although it will be shown that for linear dynamical systems, F.R. is homogeneous solution independent, it is homogeneous solution dependent in general.

Definition 3

A trajectory y(t) in the output function space is said to be an attainable trajectory from a given homogeneous solution $\varphi(\underline{x}_o, 0, t)$ if for any $\tau > 0$, there is a bounded control $\underline{u}$ such that $\varphi(\underline{x}_o, \underline{u}, t) = \underline{y}(t)$ for all t in $[0,\tau]$.

It will be seen that there are many special properties for linear dynamical systems. Now, considering a linear dynamical system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$, it is known that the solution is of the form [19]

$$\underline{y}(t) = \underline{c}\Phi(t)\underline{x}_o + \underline{c}\int_o^t \Phi(t-\tau)\underline{B}\ \underline{u}(\tau)\ d\tau + \underline{D}\ \underline{u}(t)$$

where $\Phi(t)$ is the fundamental matrix of the system and $\underline{x}_o$ is the given initial state.

Now, it is easy to see that $\underline{y}(t)$ is attainable from $\varphi(\underline{x}_o, \underline{0}, t) = \underline{c}\ \Phi(t)\underline{x}_o$ if and only if $E(t) \triangleq \underline{y}(t) - \underline{c}\ \Phi(t)\underline{x}_o$ is attainable from $\varphi(\underline{0}, \underline{0}, t) = \underline{0}$. This leads to the following theorem.

Theorem 1. A linear time-invariant dynamical system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is F.R. if and only if the zero homogeneous solution $\varphi(\underline{0}, \underline{0}, \underline{t}) = \underline{0}$ is F.R.

<Proof>: The only if part is obvious.

To show the if part, assume that $\varphi(\underline{0}, \underline{0}, \underline{t}) = \underline{0}$ is F.R. and assume the system S is not F.R. This means that there is a homogeneous solution $\varphi(\underline{x}_0^*, \underline{0}, t)$ not F.R. By the linear property of linear system, we can claim that for every $\epsilon > 0$, there exists a trajectory $\underline{y}_\epsilon(t)$

$$\|\underline{y}_\epsilon^*(t) - \varphi(\underline{x}_0^*, \underline{0}, t)\|_p \le \epsilon$$

which is not attainable from $\varphi(\underline{x}_0^*, \underline{0}, t)$.

Then, by the above observation, this implies that for any $\epsilon > 0$, there exists $E_\epsilon^*(t) \triangleq \underline{y}_\epsilon^*(t) - \varphi(\underline{x}_0^*, \underline{0}, t)$ which is not attainable from $\varphi(\underline{0}, \underline{0}, t) = 0$. This contradicts the assumption of the F.R. of $\varphi(\underline{0}, \underline{0}, t) = \underline{0}$.

The following theorem gives an intuitive explanation for F.R. in the case of linear dynamical system.

Theorem 2. A linear time-invariant system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is F.R. if and only if any bounded p-times differentiable trajectory $\underline{y}(t)$ is attainable from the zero homogeneous solution $\varphi(\underline{0}, \underline{0}, \underline{t}) = \underline{0}$.

<Proof>: The if part is trivial. Because that implies that $\varphi(\underline{0}, \underline{0}, t) = \underline{0}$ is F.R. and, by Theorem 1, implies the system S is F.R.

To show the only if part, assume that there is some $\underline{y}^*(t)$, $\|\underline{y}^*(t)\|_p = M < \infty$, which is not attainable from $\varphi(\underline{0}, \underline{0}, t) = \underline{0}$, then it can be seen that the class of all trajectories $\epsilon\, \underline{y}^*(t)$, for any $\epsilon > 0$, are not attainable, by using the linear property of linear system.

Since ϵ is arbitrary, this means in any neighborhood of $\varphi(\underline{0}, \underline{0}, t)$, there is always some trajectory not attainable. Hence, the zero homogeneous solution is not F.R. and the system is not F.R. This is a contradiction, so any p-times differentiable trajectory must be attainable from $\varphi(\underline{0}, \underline{0}, t)$.

By Theorem 1 and Theorem 2, the corollary follows:

Corollary 1: Given a linear time-invariant dynamical system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$, if there exists a homogeneous solution $\varphi(\underline{x}_o, \underline{0}, t)$ not F.R., then there exists no other homogeneous solution which could be F.R.

The following theorem will clarify a question that is: if a linear time-invariant dynamical system is not F.R. on $[0,\tau]$, is it possible to be F.R. on a smaller time interval? Here the F.R. on $[0,\tau]$ is a trivial extension of the Definition 1 in V-1.

Theorem 3: If $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is not F.R. on some time interval $[0,\tau]$, then it is not F.R. on any other time interval.

<Proof>: It is obvious that S cannot be F.R. on $[0,t]$ $[0,\tau]$.

Two other cases are considered:

(i) $[0,\tau'] \subset [0,\tau]$

Suppose S is F.R. on $[0,\tau']$. Since there exists a finite integer M, such that $M\tau' \geq \tau$, and the system S is time-invariant. S should be F.R. on $[\tau', 2\tau']$, $\cdots$, $[(M-1)\tau', M\tau']$ and hence should be F.R. on $[0,\tau]$.

This leads to a contradiction.

(ii) $[a,b] \cap [0,\tau] = \varphi$

Using the time-invariant property of S, this can be easily proved.

V-2. Rational Function Space with Rational Function Field and Frequency Domain Criteria

Since the entries of the transfer function matrix $\underline{H}(s)$ are (proper) rational functions, it would be sufficient to consider the space of rational functions in the frequency domain for the purpose of F.R. discussion. One can see an infinite-dimensional problem arises, if the usual real field R is used. However, we are going to discuss and justify the use of the set of all rational functions as a field associated with the vectors of rational functions and then can reduce the infinite-dimensional problem to a finite-dimensional problem, and therefore we can discuss the problems in a finite-dimensional vector space.

Let

$$Fr \triangleq \{f(s) \mid f(s) \text{ is a rational function in complex } s\}$$

where by rational function we mean a function of the form $p(s)/q(s)$, where $p(s)$ and $q(s)$ are polynomials in s.

Since the sum and product of rational functions are rational functions and the reciprocal of a rational function is well-defined and is also a rational function, the lemma follows.

Lemma 1: Fr is a field with addition and multiplication defined pointwisely as follows

$$(f + g)(s) = f(s) + g(s)$$

$$(f \cdot g)(s) = f(s) \cdot g(s) \qquad \text{for all } f \text{ and } g \text{ in } Fr$$

Now, let V^n denote the set of all n-tuples of rational functions, i.e.

$$V^n = \{\underline{v}(s) \mid \underline{v} \text{ is an n-tuple; each entry } v_i(s) \in Fr,\ i = 1,2,\cdots,n\}$$

If the set of real numbers is chosen as the field associated with V^n, an infinite-dimensional problem arises. However, if we choose Fr as the field, i.e. consider rational functions f(s) as scalars, the infinite-dimensional problem can be reduced to a finite-dimensional one.

<u>Theorem 1</u>: Considering elements v(s) of V^n as vectors, and elements f(s) of Fr as scalars, and defining vector addition and scalar multiplication pointwisely, the set V^n, with the operations so defined, is an n-dimensional vector space over the field Fr, which is denoted by (V^n, Fr).

<<u>Proof</u>>: It is easy to see that (V^n, Fr) is a valid vector space.

It is n-dimensional because each $\underline{v}(s)$ in V^n can be expressed as a linear combination of the n unit vectors $\underline{e}_1, e_2, \cdots, \underline{e}_n$, i.e.

$$\underline{v}(s) = \begin{bmatrix} v_1(s) \\ v_2(s) \\ \vdots \\ v_n(s) \end{bmatrix} = v_1(s)\ \underline{e}_1 + v_2(s)\ \underline{e}_2 + \cdots + v_n(s)\ \underline{e}_n$$

where

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Note that the scalars are elements of the rational function field Fr, and therefore $\underline{e}_1$, $\underline{e}_2$, $\cdots$, $\underline{e}_n$ form a basis.

Now, consider the linear time-invariant dynamical system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ and restrict our interest to the Laplace transformable inputs and outputs. As shown in Theorem 1, V-1, one can assume $\underline{x}_o = \underline{0}$ without loss of generality, and the input and output are related by

$$\underline{y}(s) = \underline{H}(s)\ \underline{u}(s)$$

where $\underline{H}(s)$ is the transfer function matrix and can be considered as a linear operator from input function space to output function space.

The following theorem which was used by Brockett and Mesarovic [2] without formal justification is to be justified from the formalization of the n-dimensional vector space (V^n, Fr).

Theorem 2: A given linear time-invariant dynamical system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is F.R. if and only if the $n \times m$ transfer function matrix $\underline{H}(s)$ is of rank n subject to the rational function field Fr.

<Proof>: Since all the entries of $\underline{H}(s)$ belong to Fr, $\underline{H}(s)$ is of rank n means there does not exist a vector $\underline{k}(s)$ in (V^n, Fr), $\underline{k}(s) \neq 0$, such that

$$\underline{k}'(s)\ \underline{H}(s) = \underline{0} \qquad \text{for all } s$$

From linear algebra, we know that for such a matrix $\underline{H}(s)$, there exists a right inverse [21], $\underline{H}_R^{-1}(s)$ such that

$$\underline{H}(s)\ \underline{H}_R^{-1}(s) = \underline{I}_n$$

Hence for any bounded, p-times differentiable $\underline{y}(s)$ in the output space, there always exist a bounded $\underline{u}(s)$ in the input space such that

$$\underline{y}(s) = \underline{H}(s)\ \underline{u}(s)$$

More explicitly, $\underline{u}(s) = \underline{H}_R^{-1}(s)\ \underline{y}(s)$ is a valid control. This means $\varphi(\underline{0}, \underline{0}, \underline{t}) = \underline{0}$ is F.R., and by Theorem 1, IV-1, the system S is F.R.

The converse is also true, since from finite-dimensional vector space theory, we know that if $\underline{H}(s)$ has rank less than n, the number of independent columns would be less than n and thus there exist some $\underline{y}(s)$ not attainable from $\varphi(\underline{0}, \underline{0}, t) = \underline{0}$. This is contradiction, and the theorem is proved.

Now, we are trying to define inner product on this vector space. Although Fr is not an ordered field, a <u>generalized inner</u> product $\langle\cdot,\cdot\rangle$ can still be defined as

$$\langle\cdot,\cdot\rangle:\quad (V^n,Fr) \times (V^n,Fr) \rightarrow Fr$$

which satisfies the following properties for every $\underline{x}$, $\underline{y}$ and $\underline{z}$ in (V^n,Fr)

(a) $\langle\underline{x} + \underline{y}, \underline{z}\rangle = \langle\underline{x}, \underline{z}\rangle + \langle\underline{y}, z\rangle$

(b) $\langle a\ \underline{x}, \underline{z}\rangle = a \langle\underline{x}, \underline{z}\rangle$ for all $a \in Fr$

(c) $\langle\underline{x}, \underline{y}\rangle = \langle\underline{y}, \underline{x}\rangle$

(d) $\langle\underline{x}, \underline{x}\rangle =$ only if $\underline{x} = \underline{0}$

It follows that for every $\underline{u}(s)$ and $\underline{v}(s)$ in (V^n, Fr), $<\underline{u},\underline{v}> = \underline{u}'(s)\ \underline{v}(s)$ is a valid generalized inner product.

Now, recall that if L is a linear operator which maps an inner product space X to another inner product space Y, the operator L* is called the <u>adjoint operator</u> of L provided that it exists and satisfies

$$L^* : Y \rightarrow X$$

and

$$<\underline{y}, L\underline{x}>_y = <L^*\underline{y}, \underline{x}>_x \qquad \text{for all } \underline{x} \text{ in } X,\ \underline{y} \text{ in } Y$$

Furthermore, if X and Y are finite-dimensional, then R[L] = R[LL*], where R[L] is the range of L.

Besides, if L is an n x m matrix which maps an m-dimensional inner product space X to an n-dimensional inner product space Y, and the inner products are defined as

$$<\underline{u}, \underline{v}> = \underline{u}'\underline{v} \qquad \text{for all } \underline{u}, \underline{v} \text{ in X or Y}$$

then

L* exists and is equal to L'. [22]

Now, assuming that $\underline{H}(s)$ maps (V^m, Fr) to (V^n, Fr), it follows that

$$\underline{H}^*(s) = \underline{H}'(s)$$

and

$$R[\underline{H}(s)] = R[\underline{H}(s)\underline{H}'(s)]$$

and therefore $\underline{H}(s)$ is of rank n if and only if the n x n matrix $\underline{H}(s)\ \underline{H}'(s)$ is of rank n. From the finite-dimensional theory, the criterion follows.

Corollary 1. (Frequency Domain Criterion)

A linear time-invariant system S is F.R. if and only if det.$[\underline{H}(s)\ \underline{H}'(s)] \not\equiv 0$ except finitely many s.

One can see that the problem of F.R. arises only when the system has multiple outputs. This is because for single output system rank $\underline{H}(s) = 1 = n$ always. Besides, a necessary condition for F.R. is that the number of outputs cannot exceed the number of inputs, that is n must be less than or equal to m.

In practical design work, $\underline{H}(s)$ is usually given instead of $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$; therefore for small dimension matrix $\underline{H}(s)$, the frequency domain criterion is rather feasible and easy for hand calculation.

An interesting criterion [2] for output controllability which is parallel to the criterion for F.R. in that the system S is output controllable if and only if there does not exist a constant vector $\underline{\lambda}$ such that

$$\underline{\lambda}'\underline{H}(s) = \underline{0} \qquad \text{for all } s$$

Hence, output controllability is a necessary condition for F.R.

V-3. Rank and Basis of the Transfer Function Matrix

In previous works, only the F.R. was emphasized and discussed. Actually since many systems are not F.R., in order to study the capability or functional attainability of a given system, one has to find out the rank and the column basis of the transfer function matrix $\underline{H}(s)$.

Before discussing the details, the following observation is made:

If the $n \times m$ transfer matrix $\underline{H}(s)$ is of rank $n-\ell$, where $\max.(o,n-m) \le \ell < \min.(n,m)$, then the rank of the $(p+1)n \times (2p+1)m$ constant matrix M_p is greater or equal to $(n-\ell)(p+1)$, where

$$M_p = \begin{bmatrix} \underline{D} & \underline{CB} & \underline{CAB} & \cdots & \underline{CA}^{p-1}\underline{B} & \underline{CA}^{p}\underline{B} & \cdots & \underline{CA}^{2p-1}\underline{B} \\ \underline{0} & \underline{D} & \underline{CB} & \cdots & \underline{CA}^{p-2}\underline{B} & \underline{CA}^{p-1}\underline{B} & \cdots & \underline{CA}^{2p-2}\underline{B} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \underline{0} & \underline{0} & \underline{0} & \cdots & \underline{D} & \underline{CB} & \cdots & \underline{CA}^{p-1}\underline{B} \end{bmatrix}$$

and was used as a criterion matrix for F.R. by Brockett and Mesarovic [2] in such a way that $\underline{H}(s)$ is of rank n if and only if M_p is of rank $(p+1)n$.

This is true because if $\underline{H}(s)$ is of rank $n-\ell$, then there exists an $(n-\ell) \times m$ sub matrix $\underline{H}_{sub}(s)$ with linearly independent rows. Correspondingly, there exist submatrices $\underline{C}_{sub}$ and $\underline{D}_{sub}$, such that

$$\underline{H}_{sub}(s) = \underline{C}_{sub}\,(IS-\underline{A})^{-1}\,\underline{B} + \underline{D}_{sub}$$

and therefore, by Brockett and Mesarovic's criterion, the submatrix $\bar{M}_p$ of M_p which is generated by $\underline{A}$, $\underline{B}$, $\underline{C}_{sub}$ and $\underline{D}_{sub}$ has rank $(n-\ell)(p+1)$. Thus, the rank of M_p is greater or equal to $(n-\ell)(p+1)$.

From this observation, the following statement can be made:

If the rank of M_p is less than or equal to $(n - \ell)(p + 1)$, then the rank of $\underline{H}(s)$ is not greater than $n - \ell$.

This provides us with an estimate (upper bound) of the rank of $\underline{H}(s)$ which is helpful when the system is of large size.

The procedure for finding the rank and basis of $\underline{H}(s)$ could be as follows:

1. If the system size is small and if the transfer function matrix is given or is easy to calculate, use the frequency domain approach. More specifically, if $n > m$, the rank of $\underline{H}(s)$ can not exceed m and is obviously less than n. Thus, check the linear independency of the column vectors of $\underline{H}(s)$ by calculating det $[\underline{H}'(s)\ \underline{H}(s)]$ to see if it is zero or not. If not, the rank of $\underline{H}(s)$ is m. (Note that this criterion is different from that in V-2 which was used for row independence check). If it vanishes, use the same criterion to check the sub-matrices $\underline{H}_{sub}(s)$ which are obtained by deleting one column each time, until one submatrix with independent columns is found. Then the rank of $\underline{H}(s)$ is equal to the number of columns of this sub-matrix which is also to be the <u>basis matrix</u> of $\underline{H}(s)$.

If $n < m$, use the criterion in V-2 to check the rank of $\underline{H}(s)$. If it equals n, stop. If not, use the criterion det $[\underline{H}'(s)\ \underline{H}(s)] \neq 0$ to check the linear independency of the column vectors of an $n \times (n-1)$ sub-matrix $\underline{H}_{sub}(s)$ of $\underline{H}(s)$ and so on.

2. If the system size is too large for hand calculation, use any of the parameter matrix criteria [2, 6, 14, 16, 17]. It is suggested to find an upper bound for the rank of $\underline{H}(s)$ first, say b, using

the above observation. Then the basic principle is as before, except that corresponding to the deletion of columns of $\underline{H}(s)$, there are two $n \times b$ sub-matrices $\underline{B}_{sub}$ and $\underline{D}_{sub}$ by deleting corresponding columns of $\underline{B}$ and $\underline{D}$ such that

$$\underline{H}_{sub}(s) = \underline{C}(IS-\underline{A})^{-1}\,\underline{B}_{sub} + \underline{D}_{sub}$$

Then any parameter matrix criterion can be used. The procedure will stop when first time a submatrix $\underline{H}_{sub}(s)$ (or $\underline{B}_{sub}$, $\underline{D}_{sub}$) with independent columns is found, and that matrix will be called the basis matrix of $\underline{H}(s)$ which is to serve as a generator for attainable outputs, and is denoted by $\underline{H}_B(s)$.

V-4. Rational Functional-attainable Set

In this section, the attainable rational output functions will be characterized, but not all the rational functions in the range of $\underline{H}(s)$ can be attained. First of all, the concept of admissible input and output functions is introduced.

Definition 1

A rational input function $\underline{u}(s)$ for a linear time-invariant dynamical system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is said to be admissible, if each element of $\underline{u}(s)$ is a proper rational function or zero, i.e. the degree of the denominator polynomial is greater than the degree of the numerator polynomial.

A rational function in the range of $\underline{H}(s)$ is said to be admissible, if it is generated by some admissible input $\underline{u}(s)$.

The purpose of this restriction is to exclude the possibility of impulse input functions (for instance, the delta function in the frequency domain is equal to 1 which is not proper rational) and the admissible input function so defined is bounded in any finite time interval.

By def. 3 in V-1, the rational functional-attainable set $\underline{A}_r(\underline{X}_o)$, for given initial state $\underline{X}_o$, is

$$\underline{A}_r(\underline{X}_o) \triangleq \left\{ \underline{y}(s) \,|\, \underline{y}(s) = \varphi(\underline{x}_o, \underline{u}, t), \text{ some } \underline{u} \in (V^m \cdot Fr) , \atop \underline{u} \text{ is admissible} \right\}$$

On the other hand, the set of all rational functions which are generated by some rational input function subject to the initial condition $\underline{x}_o$, is the set

$$V_r(\underline{x}_o) \triangleq \underline{C}(IS-\underline{A})^{-1} \underline{x}_o + R[\underline{H}_B(s)]$$

which is a linear variety (manifold) in (V^n, Fr).

It should be noted that the set of all proper rational functions is not a field, since the reciprocal of a proper rational is not proper any more, and the set of all proper rational $\underline{u}(s)$ in (V^m, Fr) is not a subspace. Thus, the rational functional-attainable set $\underline{A}_r(\underline{x}_o)$ cannot be a subspace but a subset embedded in $V_r(\underline{x}_o)$.

Without loss of generality, assume $\underline{x}_o = 0$, then $V_r(\underline{0})$ is a subspace in (V^n, Fr) and the whole space can be decomposed as the direct sum of $V_r(\underline{0})$ and its orthogonal complement $V_r^{\perp}(\underline{0})$, i.e.

$$V^n = V_r(\underline{0}) \oplus V_r^{\perp}(\underline{0}) \ .$$

Any rational $\underline{y}(s)$ in (V^n, Fr) can be decomposed into two orthogonal parts, i.e.

$$\underline{y}(s) = \underline{y}_r(s) + \underline{y}_r^{\perp}(s)$$

where $\underline{y}_r(s)$ in $V_r(\underline{0})$ and $\underline{y}_r^{\perp}(s)$ in $V_r^{\perp}(\underline{0})$, and $\underline{y}_r^{\perp}(s)$ is called the unreachable component. Any (admissible) rational $\underline{y}(s)$ is attainable if and only if it contains no unreachable component.

V-5. Output Trajectory Test

If the system is not F.R. in the full sense of the term, the following question is of interest:

Is a desired output trajectory $\underline{y}^*(t)$ attainable by the linear time-invariant system S from a homogeneous solution $\varphi(\underline{x}_o, \underline{0}, t)$?

To answer this question, it is necessary to:

1. Define $E(s) \triangleq \underline{y}^*(s) - \underline{C}(IS-\underline{A})^{-1}\, \underline{x}_o$ and add $E(s)$ to the basis matrix $\underline{H}_B(s)$ to form an augmented matrix $\underline{H}_A(s)$

$$\underline{H}_A(s) \triangleq [E(s) \vdots \underline{H}_B(s)]$$

2. Check the linear dependency of the columns of $\underline{H}_A(s)$ by $\det[\underline{H}_A'(s)\ \underline{H}_A(s)] = 0$?

If yes, then $\underline{y}^*(s)$ is attainable from $\varphi(\underline{x}_o, \underline{0}, t)$. Otherwise, it is not.

If the size of the system is too large for hand calculation, use any standard canonical realization [19] to find out $[\underline{A}_a, \underline{B}_a, \underline{C}_a, \underline{D}_a]$ such that

$$\underline{H}_A(s) = \underline{C}_a(IS-\underline{A}_a)^{-1}\,\underline{B}_a + \underline{D}_a$$

Since each elements of $\underline{H}_A(s)$ is (proper) rational, the realizability of $\underline{H}_A(s)$ is guaranteed.

Then any parameter matrix criterion can be applied to check the linear dependency of the columns of $\underline{H}_A(s)$.

CHAPTER VI

INPUT FUNCTIONAL OBSERVABILITY AND DUALITY

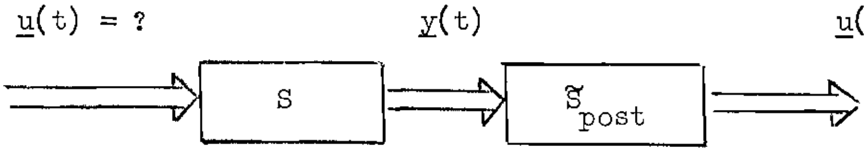
It arises in many cases, that the only measurable quantity from a dynamic system is the output function. However, frequently, we would like to know the non-measurable input function $\underline{u}(t)$ from the knowledge of the observed output $\underline{y}(t)$. For instance, this can happen in the information recovery problem in coding theory, in adaptive filtering and control, and may play an important role in the diagnostic techniques both in engineering and bio-medical application in the near future.

The capability of uniquely determining the input function from the knowledge of the output function is called the <u>input functional observability</u> (will be abbreviated as I.F.O. in the sequel) which is another term for invertibility in other literatures.

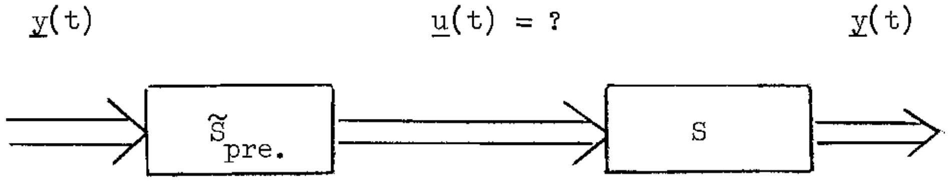
Hence, if a given system S is I.F.O., then we can design an inverse system $\tilde{S}$, post-cascaded with S such that the output of $\tilde{S}$ would be the unknown input of S [6].

However, there have seemed to have two different types of inverse being used in the literature [6].

One is the inverse mentioned above, called the post-inverse system and is referred to as <u>inverse system</u> in this work. In diagram, i.e.



Another kind is called pre-inverse which is pre-cascaded with the given system S such that we can generate any admissible output function $\underline{y}(t)$ of S by feeding the signal $\underline{y}(t)$ into $\tilde{S}_{pre}$. In diagram, i.e.



The existence of pre-inverse is just the functional reproducibility and the pre-inverse system is referred to as <u>function generator</u> in this work.

<u>Definition 1</u>:

A linear time-invariant system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is said to be input functionally observable (I.F.O.) in time interval $[t_o, t_1]$ if and only if we can exactly recognize the input function $\underline{u}(t)$ over $[t_o, t_1]$ by the knowledge of output function $\underline{y}(t)$ over $[t_o, t_1]$ and the initial state $\underline{x}_o$.

This definition is also valid for time-varying systems. We are going to prove that I.F.O. is time interval independent for time-invariant system. Without loss of generality, let us assume that $\underline{x}_o = \underline{0}$, then the input $\underline{u}(t)$ and output $\underline{y}(t)$ are related by

$$\underline{y}(t) = \int_{t_o}^{t_1} \underline{C}\ \Phi(t-\tau)\underline{u}(\tau)\ d\tau + \underline{D}\ \underline{u}(t) \qquad t \in [t_o, t_1]$$

Define the linear operator L by

$$(L\,\underline{u})(t) \triangleq \int_{t_o}^{t_1} \underline{C}\;\Phi(t-\tau)\;\underline{u}(\tau)\;d\tau + \underline{D}\;\underline{u}(t)$$

$$= \underline{y}(t) \qquad\qquad t \in [t_o t_1]$$

Now, if we cannot exactly recognize the input $\underline{u}(t)$ by observing $\underline{y}(t)$ over $[t_o t_1]$, that means there exist $\underline{u}_1(t)$ and $\underline{u}_2(t)$, $\underline{u}_1(t) \not\equiv \underline{u}_2(t)$ almost everywhere (i.e. $\underline{u}_1(t)$ can be equal to $\underline{u}_2(t)$ only on a set of measure o) such that

$$L\,\underline{u}_1 = L\,\underline{u}_2$$

$$= \underline{y}$$

or equivalently, there exists an $\underline{u}_o \not\equiv \underline{0}$ almost everywhere (a.e.)

$$\underline{u}_o \triangleq \underline{u}_1 - \underline{u}_2$$

such that

$$L\,\underline{u}_o = \underline{0}$$

Thus we can make the following definition.

Definition 2:

Given a linear time-invariant system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$, an input function $\underline{u}(t) \not\equiv \underline{0}$ a.e. is said to be non-deducible over $[t_o t_1]$ iff

$$(L\,\underline{u})(t) = \underline{0} \qquad \forall \quad t \in [t_o t_1]$$

The following theorem follows trivially.

Theorem 1: A linear time-invariant System S is I.F.O. over $[t_o, t_1]$ if and only if there is no non-deducible input function over $[t_o, t_1]$.

Now, we are going to prove that I.F.O. is time independent for linear time-invariant system.

Theorem 2: A linear time-invariant system S is I.F.O. on [c,d] if and only, if it is I.F.O. on [a,b], where $[a,b] \subset [c,d]$.

<Proof>: Suppose S were not I.F.O. on [a,b] then, by Theorem 1, there exists a $\underline{u}_o(t)$, $t \in [a,b]$ such that

$$(L\,\underline{u}_o)(t) = \underline{0} \qquad t \in [a,b]$$

Thus for any given $\underline{y}(t)$ on [c,d], the $\underline{u}(t)$ such that

$$\underline{y}(t) = (\underline{L}\,\underline{u})(t) \qquad t \in [c,d]$$

is not unique, since there is another $\underline{u}'(t)$

$$\underline{u}'(t) \triangleq \begin{cases} \underline{u}(t) + \underline{u}_o(t) & t \in [a,b] \\ \underline{u}(t) & t \in [c,d] - [a,b] \end{cases}$$

such that

$$(L\,\underline{u}')(t) = \underline{y}(t) \qquad \text{on } [c,d]$$

On the other hand, using the time-invariant property of time-invariant system, we can see that S is I.F.O. on [a,b] implies that S is I.F.O. on [c, b-a+c].

Then, repeating the argument M times, where $M(b-a) \geq (d-c)$, we can conclude that S is I.F.O. on [c,d] too.

This theorem says that if S is I.F.O. on [O.T] then it is I.F.O. on any sub-interval of [O.T].

Without loss of generality, in the following discussion, when we say S is I.F.O., we mean S is I.F.O. on $[0,\infty)$. Now, let us consider the frequency domain version again.

As input function $\underline{u}(s)$ is said to be "non-deducible" if and only if

$$\underline{H}(s)\ \underline{u}(s) = \underline{0} \qquad \text{for all } s$$

where $\underline{H}(s)$ is the transfer function matrix of the system S.

Theorem 3: A linear time-invariant system $S = [\underline{A}\ \underline{B}\ \underline{C}\ \underline{D}]$ is I.F.O. if and only if the rank of the $n \times m$ transfer function matrix $\underline{H}(s)$ is m. (over the rational function field Fr)

<Proof>: By Theorem 1 and Theorem 2, we know that S is I.F.O. if and only if there exists no input function $\underline{u}(s)$ such that

$$\underline{H}(s)\ \underline{u}(s) = \underline{0} \qquad \text{for all } s$$

Since the elements of $\underline{H}(s)$ are (proper) rationals, this is equivalent to that $\underline{H}(s)$ is of rank m over the rational function field Fr, or, alternately, the column vectors of $\underline{H}(s)$ are linearly independent.

Corollary 1: The linear time-invariant system S is I.F.O. if and only if

$$\det.[\underline{H}'(s)\ \underline{H}(s)] \neq 0 \qquad \text{except finitely many } s$$

In contrast to the case of F.R., a necessary condition of I.F.O. is that the number of input variables cannot exceed the number of output variables, (i.e. $n \geq m$). Thus, a system can be both F.R. and I.F.O., only if it has equal numbers of input variables and output variables. Note that a system with only one input variable is always I.F.O.

Now, suppose that $\underline{u}(s)$ is an admissible rational function, then $\underline{u}(s)$ is said to be non-deducible if

$$\underline{u}(s) \in N[\underline{H}(s)]$$

where

$$N[H(s)] = \text{null space of } \underline{H}(s) \text{ in } (V^m, Fr).$$

By the following definition, the space (V^m, Fr) will be decomposed as the direct sum of $N[\underline{H}(s)]$ and its orthogonal complement.

Definition 3:

An (admissible) rational input function $\underline{u}(s)$ is said to be deducible if $\underline{u}(s)$ is orthogonal to the null space of $\underline{H}(s)$, i.e.

$$\underline{u}(s) \perp N[\underline{H}(s)]$$

or

$$\underline{u}(s) \in \overset{\perp}{N}[\underline{H}(s)] = \text{the orthogonal complement of } N[\underline{H}(s)].$$

The statement that $\underline{u}(s)$ is deducible actually means that $\underline{u}(s)$ contains no non-deducible component.

The following theorem will related F.A. and I.F.O. in a dual sense.

Theorem 4 (Dual): The linear time-invariant system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is F.R. if and only if the linear time-invariant adjoint system $S^* = [\underline{A}', \underline{C}', \underline{B}', \underline{D}']$ is I.F.O.

Moreover, the rational output trajectory $\underline{y}(t) = \underline{r}(t)$ is attainable from $\varphi(\underline{0}, \underline{0}, t) = \underline{0}$, by the system S, implies that the rational function $\underline{w}(t) = \underline{r}(t)$ is deducible from the output of the adjoint system S*,

$$S^* : \begin{cases} \dot{\underline{p}}(t) = \underline{A}'\underline{p}(t) + \underline{C}'\ \underline{w}(t) \\ \underline{v}(t) = \underline{B}'\underline{p}(t) + \underline{D}'\ \underline{w}(t) \end{cases}$$

$$\underline{p}(0) = \underline{0}$$

<Proof>: Note that the transfer function matrix of the adjoint system S* is

$$\begin{aligned} \underline{H}^*(s) &= \underline{B}'(IS-\underline{A}')^{-1}\,\underline{C}' + \underline{D}' \\ &= [\underline{C}(IS-\underline{A})^{-1}\,\underline{B} + \underline{D}]' \\ &= \underline{H}'(s) \end{aligned}$$

where $\underline{H}'(s)$ is the transpose of $\underline{H}(s)$ and is the adjoint operator of $\underline{H}(s)$.

By Theorem 3, the system S* is I.F.O. if and only if the $m \times n$ transfer matrix $\underline{H}^*(s)$ is of rank n. This is equivalent to that the $n \times m$ matrix $\underline{H}(s)$ is of rank n which is a necessary and sufficient condition (Theorem 2, V-2) for the system S being F.R.

With regard to the second part of the theorem, a rational trajectory $\underline{y}(t) = \underline{r}(t)$ is attainable from $\varphi(\underline{0}, \underline{0}, t) = \underline{0}$, by the System S, if and only if

$$\underline{y}(s) \in R[\underline{H}(s)]$$

and $\underline{y}(s) = \underline{r}(s)$ is admissible

which implies that

$$\underline{y}(s) = \underline{r}(s) \in N^{\perp}[\underline{H}'(s)]$$

Thus, by Definition 3, $\underline{r}(t)$ is attainable by S implies $\underline{r}(t)$ is deducible by S*.

Note that due to the restriction of admissibility, the converse of the second part of the theorem does not hold.

CHAPTER VII

INVERSE SYSTEMS

As mentioned in VI, the motivation of the study of invertibility (I.F.O.) is to see if we can find out the unknown input from the measurement of output. However, this purpose can be achieved by using a device called "inverse system" provided that the given linear dynamic system is I.F.O.

There have been many algorithms for inverse systems design [14, 16, 18]. The common characteristics of those algorithms are the use of system coefficient matrices $\underline{A}$, $\underline{B}$, $\underline{C}$ and $\underline{D}$, such that they are feasible for large-sized system calculation.

However, we are going to solve this problem by frequency-domain approach which is feasible for many practical design work and for small-sized hand calculation, and is easier to understand.

Suppose S is I.F.O., and assume:

(i) $\underline{H}_{(s)}$ is a square matrix $(n = m)$:

The inverse system $\underline{\tilde{H}}_{(s)}$ is just $\underline{H}^{-1}{}_{(s)}$, and the unknown input $\underline{u}_{(s)}$ can be obtained by post-cascading $\underline{\tilde{H}}_{(s)}$ with $\underline{H}_{(s)}$.

(ii) $\underline{H}_{(s)}$ is a rectangular matrix $(n \geq m)$:

Since S is I.F.O., by Theorem 3, IV-B, the rank of $\underline{H}_{(s)}$ is m. Hence, the left inverse of $\underline{H}_{(s)}$ exists but not unique [21]. One of the left inverse is

$$\underline{H}_L{}^{-1}{}_{(s)} = (\underline{H}'_{(s)}\, \underline{H}_{(s)})^{-1}\, \underline{H}'_{(s)}$$

such that

$$\underline{H}_L^{-1}(s)\ \underline{H}(s) = I_m$$

Thus we can use $\underline{H}_L^{-1}(s)$ as the inverse system $\tilde{\underline{H}}(s)$.

Another method is to find out m linearly independent rows from $\underline{H}(s)$, and then we get an $m \times m$ non-singular sub-matrix of $\underline{H}(s)$. The inverse matrix of this sub-matrix can be an inverse system of $\underline{H}(s)$.

Sometimes, the given system S is not I.F.O. owing to the fact that the number of input variables is greater than the number of output variables. But, if some of the input variables are known, then their effects can be subtracted from the output, and possibly the reduced system could be I.F.O.

One important question is that is the inverse system a dynamical system?

The question can be considered by two cases:

(i) The system is of the form $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$, $\underline{D} \neq \underline{0}$:

The answer is that the inverse system $\tilde{S}$ is a dynamical system, and is of the same form as S, i.e.

$$\tilde{S} = [\tilde{\underline{A}}, \tilde{\underline{B}}, \tilde{\underline{C}}, \tilde{\underline{D}}] .$$

(iii) The system is of the form $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$, but $\underline{D} = \underline{0}$:

The answer is that the inverse system $\tilde{S}$ is not a dynamical system but a dynamical system plus a number of ideal differentiators. We call such kind of systems hybrid-dynamical systems [6].

If we do not use ideal differentiators, then the output of the inverse system would be the L-th integral of the unknown input function, where L is some integer less than p. This is the so called L-integral inverse by Sain and Massey [18].

This can be easily seen by considering the following single-variable system:

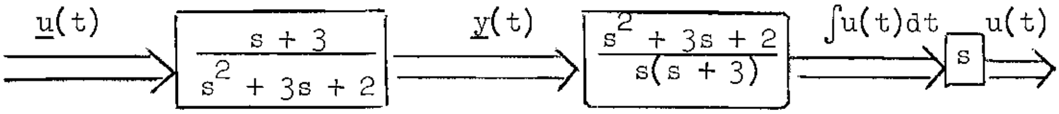
Suppose

$$\underline{H}(s) = \frac{s + 3}{s^2 + 3s + 2}$$

then

$$\tilde{\underline{H}}(s) = \underline{H}^{-1}(s) = \frac{s^2 + 3s + 2}{s + 3} = s \cdot \frac{s^2 + 3s + 2}{s^2 + 3s}$$

i.e.



This is because the elements of the transfer function matrix of a dynamical system of this type $S = [\underline{A}, \underline{B}, \underline{C}]$, are proper rational functions, while the reciprocal of a proper rational function is not proper, and the degree of the numerator polynomial is greater than the degree of the denominator polynomial. Hence, differentiator must be involved and the inverse system $\tilde{S}$ cannot be a dynamical system.

CHAPTER VIII

DESIGN AND APPLICATIONS

VIII-1. Feedback Compensation

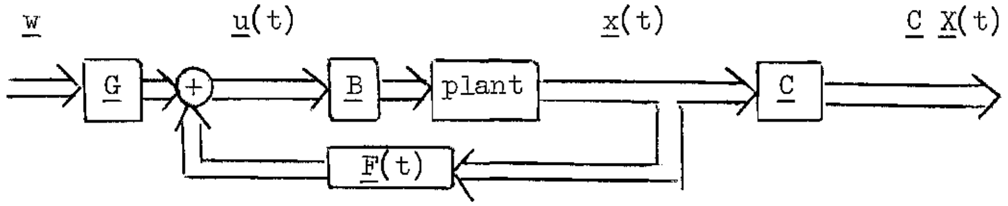
Feedback is a very powerful concept in automatic control. It can be used in many different ways to modify the performance of a given system or to force the system to do some desired job automatically. For instance, it is possible to use feedback to change the location of system poles, to stabilize an unstable system, to decouple a multi-variable system [7] and to form an optimal control and filtering system [1], etc.

It is natural to ask the question: can feedback improve the functional attainability? Or can we make a non-F.R. system F.R. by using suitable feedback?

Before stating the theorem which will answer the question, let us classify different types of feedback:

(i) Static Feedback:

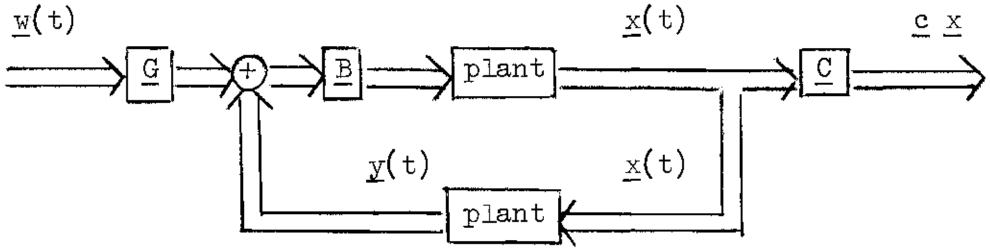
$$\underline{u}(t) = \underline{F}(t)\ \underline{x}(t) + \underline{G}\ \underline{w}(t)$$



If $\underline{F}(t)$ is constant, then it is a time-invariant feedback.

(ii) Dynamical Feedback:

$$\begin{cases} \underline{u}(t) = \underline{v}(t) + \underline{G}\ \underline{w}(t) \\ \dot{\underline{v}}(t) = \underline{f}(\underline{v}, \underline{x}) \end{cases}$$



Theorem 1: A linear time-invariant dynamical system S is F.R. if and only if any (time-invariant) feedback closed-loop system S_B, with input matrix $\underline{G}$ having a right inverse, is F.R.

<Proof>: Suppose S is F.R. and let $\underline{u}^o$ be the executive input which accomplishes a desired admissible transfer.

If static feedback is used, then the same transfer can be done by the input $\underline{w}(t)$.

$$\underline{w}(t) = \underline{G}^{-1}[\underline{u}^o(t) - \underline{F}\ \underline{x}(t)]$$

where $\underline{G}^{-1}$ is the right inverse of $\underline{G}$.

Similar argument applies to dynamical feedback. Thus, the closed-loop system S_B is F.R.

On the other hand, if S_B is F.R., and $\underline{w}^o$ is the executive input, the same transfer can be done by

$$\underline{u}(t) = \underline{F}\,\underline{x}^o(t) + \underline{G}\,\underline{w}^o(t)$$

where $\underline{x}^o(t)$ is the corresponding closed-loop state vector. Thus, S is F.R. also.

It is easy to see that the theorem also holds for time-varying feedback. An important result comes from this theorem, that is: if the given linear system S is not F.R., then no closed-loop feedback compensated system S_B could be F.R. This excludes the possibility of compensating a system to be F.R. solely by feedback rather than modifying the input matrix $\underline{B}$ or $\underline{D}$.

For I.F.O., a weaker statement can be made.

Theorem 2. (Dual)

A linear time-invariant system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is I.F.O., if and only if the linear system $\bar{S} = [\underline{A}, \underline{B}, \bar{\underline{C}}, \bar{\underline{D}}]$ is I.F.O., where $\bar{\underline{C}} = \underline{K}\,\underline{C}$, $\bar{\underline{D}} = \underline{K}\,\underline{D}$, and $\underline{K}$ has left inverse.

<Proof>: By Theorem 3 in V, the system $\bar{S}$ is I.F.O. if and only if the system $\bar{S}^* = [\underline{A}', \bar{\underline{C}}', \bar{\underline{B}}, \bar{\underline{D}}]$ is F.R.

Note that

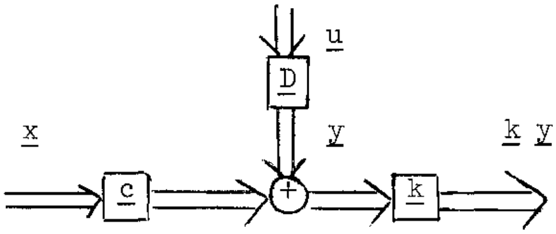
$$\bar{\underline{C}}' = \underline{C}'\underline{K}'$$

$$\bar{\underline{D}}' = \underline{D}'\underline{K}'$$

and $\underline{K}'$ has right inverse.

Then, by the above theorem (Theorem 1, VIII-1), $S^* = [\underline{A}', \underline{C}', \underline{B}', \underline{D}']$ is F.R. By Theorem 4, VI again, this is equivalent to that $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is I.F.O. The theorem is thus proved.

An important result follows from this theorem, which excludes the possibility of making a system I.F.O. solely by re-manipulating the output (namely, post-cascading a matrix $\underline{k}$), without modifying the output matrix $\underline{C}$ or $\underline{D}$, i.e.



VIII-2. <u>Existence of F.R. and I.F.O. Systems</u>

It has been shown in the last section that the functional attainability cannot be improved by solely using feedback compensation without modifying the structures of $\underline{B}$ or $\underline{D}$, and thus it can be seen that the key for F.R. is the structure of $\underline{B}$ or $\underline{D}$.

If a given system S is not F.R., it is natural to ask the question like: are there other input matrices which make the system F.R.? or more precisely, for given matrices $\underline{A}$, $\underline{C}$, and $\underline{D}$, is there an input matrix $\underline{B}$ such that $[\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is F.R.? Since if $\underline{D}$ has right inverse, the problem becomes trivial, $\underline{D}$ will be assumed with no right inverse.

The following theorems will answer this question.

<u>Lemma 1</u>: For given matrices $\underline{A}$, $\underline{C}$, and $\underline{D}$ ($\underline{D}$ has no right inverse), there exists a matrix $\underline{B}$ such that S = $[\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is F.R. if and only if $[\underline{A}, \underline{I}, \underline{C}, \underline{D}]$ is F.R.

<<u>Proof</u>> The if part is trivial.

To show the only if part, assume $[\underline{A}, \underline{I}, \underline{C}, \underline{D}]$ is not F.R. then, by Theorem 1, VIII-1, any system $[\underline{A}, \underline{\bar{B}}, \underline{C}, \underline{D}]$ cannot be F.R., with $\underline{\bar{B}} \triangleq I\ \underline{G}$.

In particular, $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ with $\underline{B} = I\ \underline{B}$ is not F.R. This is a contradiction.

Lemma 2: The system $[\underline{A}, \underline{I}, \underline{I}, \underline{D}]$ is both F.R. and I.F.O. for any $\underline{A}$ and $\underline{D}$.

<Proof>: The lemma follows from the fact that the square matrix $(IS-\underline{A})^{-1}$ is invertible.

The following theorem will given the necessary and sufficient condition for the existence of input matrices which make the system F.R.

Theorem 1: The necessary and sufficient condition for the existence of input matrices $\underline{B}$ such that the system $[\underline{A}, \underline{B}, \underline{C}]$ is F.R., is that $\underline{C}$ has right inverse.

<Proof>: Sufficiency: Since $[\underline{A}, \underline{I}, \underline{I}]$ is F.R. and $\underline{C}$ has right inverse, the system $[\underline{A}, \underline{I}, \underline{C}]$ is F.R. By lemma 1, there exists matrices $\underline{B}$, such that $[\underline{A}, \underline{B}, \underline{C}]$ is F.R.

necessity: Assume $\underline{C}$ has no right inverse, then the row vectors of $\underline{C}$ are not linearly independent. Thus the system $[\underline{A}, I, \underline{C}]$ is not F.R., and by Lemma 1, there exists no input matrix $\underline{B}$, such that $[\underline{A}, \underline{B}, \underline{C}]$ is F.R. This leads to a contradiction.

Corollary 1: A necessary condition for a system to be F.R. is that the dimension of the state space must be greater or equal to the dimension of the output vector.

<Proof>: Assume the state space dimension p is less than the output dimension n, then the system $[\underline{A}, I, \underline{C}, \underline{D}]$ cannot be F.R. This

is because I is a p $\times$ p identity matrix and thus the input dimension is p which is less than the output dimension. By Lemma 1, this implies that there exists no input matrix B such that $[\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ could be F.R.

The following corollary gives the minimum dimension of the input vector to form a F.R. system, and realization of the input matrix other than the identity matrix I is given.

Corollary 2: Given matrices $\underline{A}$ and $\underline{C}$, then the minimum dimension of the matrix $\underline{B}$ such that $[\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ could be F.R., is p $\times$ n. An n $\times$ p sub-matrix I_{sub} of the p $\times$ p identity matrix I could be a realization.

The following corollary characterizes a class of input matrices $\underline{B}$ which make the system F.R.

Corollary 3: If the system $[\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is F.R., then any system $[\underline{A}, \bar{\underline{B}}, \underline{C}, \underline{D}]$ is F.R., provided that $\bar{\underline{B}} = \underline{B}\ \underline{G}$ and $\underline{G}$ has right inverse.

Similar considerations apply to I.F.O.

Lemma 3: For given matrices $\underline{A}$, $\underline{B}$ and $\underline{D}$ ($\underline{D}$ has no left inverse), the necessary and sufficient condition for the existence of output matrices $\underline{C}$, such that $[\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is I.F.O., is $[\underline{A}, \underline{B}, \underline{I}, \underline{D}]$ is I.F.O.

<Proof>: By Theorem 2, VIII-1 and similar argument as in Lemma 1, it can be proved easily.

Intuitively speaking, the identity matrix I can be considered as a perfect output matrix which reveals all the information of the system. The lemma states that if the unknown input cannot be exactly determined by the perfect information, then no other output matrix would be of help. The following theorem is a dual to Theorem 1.

Theorem 2: The necessary and sufficient condition for the existence of output matrices $\underline{C}$ such that the system $[\underline{A}, \underline{B}, \underline{C}]$ is I.F.O., is that $\underline{B}$ must have left inverse.

Corollary 4: A necessary condition for a given system to be I.F.O., in that the dimension of the input vector cannot exceed the dimension of the state vector, (i.e. $p \geq m$).

Corollary 5: Given matrices $\underline{A}$ and $\underline{B}$, the minimum dimension of the output matrix $\underline{C}$ such that $[\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is I.F.O., is $m \times p$. An $m \times p$ sub-matrix I_{sub} of the $p \times p$ identity matrix I can be a realization.

Similarly, for each $\underline{C}$ such $[\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is I.F.O., there exist a class of such $\underline{C}$'s which are characterized by:

Corollary 6: If $[\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ is I.F.O., then any system $[\underline{A}, \underline{B}, \underline{\bar{C}}, \underline{D}]$ is I.F.O., provided that $\underline{\bar{C}} = \underline{K}\ \underline{C}$ and $\underline{C}$ has left inverse.

VIII-3. Deisgn of a Dynamical Pseudo-inverse System

It has been seen in VII, that the inverse, $\tilde{S}_o$, of a linear-time-invariant dynamical system of this type $S_o = [\underline{A}, \underline{B}, \underline{C}]$ is not dynamical but hybrid dynamical, i.e. differentiators are involved. The obstacle for the applications of such a hybrid dynamical inverse system is due to the inaccuracy of the practically available differentiators [23].

However, it has been shown [6] that the inverse of a linear time-invariant dynamical system of this type, $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ where $\underline{D}$ has left inverse, is also a dynamical system, without differentiators. This inverse is given by $\tilde{S} = [\underline{\tilde{A}}, \underline{\tilde{B}}, \underline{\tilde{C}}, \underline{\tilde{D}}]$ where

$$\underline{\tilde{A}} = \underline{A} - \underline{B}\,\underline{D}^{-1}\,\underline{C}$$
$$\underline{\tilde{B}} = \underline{B}\,\underline{D}^{-1}$$
$$\underline{\tilde{C}} = -\,\underline{D}^{-1}\,\underline{C}$$
$$\underline{\tilde{D}} = \underline{D}^{-1}$$
$$\underline{D}^{-1} = \text{left inverse of } \underline{D}$$

Intuitively, S can approximate S_o, provided that the norm of $\underline{D}$ is small enough. Thus, it is natural to try to use $\tilde{S}$ to approximate $\tilde{S}_o$ such that we can have a dynamical inverse for S.

Some problems arise, and they are: how closely can $\tilde{S}$ approximate $\tilde{S}_o$? Or how closely does the estimated input $\hat{\underline{u}}(t)$ deviate from the true unknown input $\underline{u}(t)$? And does $\hat{\underline{u}}(t)$ converge to $\underline{u}(t)$ as we choose the norm of $\underline{D}$ smaller and smaller?

The following discussion will answer these questions, and will prove that $\tilde{S}$ is indeed a good approximation to $\tilde{S}_o$, under some reasonable assumptions.

Firstly, the following lemma is necessary.

Lemma 1. Let B[0,T] be a set of all n-vector-valued bounded functions defined on [0,T], with the sup-norm defined in V-1, i.e.

$$\|\underline{f}\| = \sup_{t\,\epsilon[0,T]} |\underline{f}(t)|$$

then B[0,T] is a Banach space, i.e. every Cauchy sequence in B[0,T] converges.

<Proof>: Let $\{\underline{f}_k\}$ be a Cauchy sequence in B[0,T], i.e. for every $\Sigma > 0$, there exists an N, such that

$$\|\underline{f}_n - \underline{f}_m\| < \Sigma \qquad \text{for all } n,\ m \geq N$$

or

$$\sup_{t\epsilon[0,T]} |\underline{f}_n - \underline{f}_m| < \Sigma \qquad \text{for all } n,\ m \geq N$$

This implies that

$$|\underline{f}_n - \underline{f}_m| < \Sigma \qquad \text{for all } t\epsilon[0,T] \text{ and } n,\ m \geq N$$

and, in particular, for fixed t, $\{\underline{f}_k(t)\}$ is a Cauchy sequency in R^n subject to the norm $|\cdot|$ as defined in V-1 which is known as ℓ_1 norm and is complete [22].

Since every Cauchy sequence converges with a complete norm, $\{\underline{f}_k(t)\}$ converges to $\underline{f}(t)$ for every $t\epsilon[0,T]$ which is equivalent to that $\{\underline{f}_k\}$ converges to $\underline{f}$ in B[0,T]. Thus the norm $\|\cdot\|$ is complete, and B[0,T] is a Banach space.

The following is a fundamental result from linear operator theory [22].

Lemma 2: Let L be a linear operator which transforms a normed linear space U to another normed linear space V, then L is continuous if and only if L is bounded.

The boundedness of L can be understood from the following definition.

Definition 1 A linear operator defined as above is said to be bounded if and only if the norm of L is bounded by a finite constant. i.e.

$$\|L\| < K$$

where the norm of L is defined as

$$\|L\| \triangleq \sup_{\|\underline{x}\| \leq 1} \frac{\|L\,\underline{x}\|}{\|\underline{x}\|}$$

The following theorem which sometimes is known as Banach Inverse Theorem [22] will play an important role in the convergence proof.

Theorem 1. Let L be a linear operator mapping a Banach space B_1 to another Banach space B_2. If L^{-1} exists and L is continuous, then L^{-1} is also continuous.

Now, some property from the linear systems theory is needed.

Lemma 3. [24]

A linear time-invariant dynamical system S is asymptotically stable implies S is bounded-input bounded-output stable (B.I.B.O.)

Intuitively speaking, B.I.B.O. stable is that for any bounded input, the output function is bounded and is equivalent to that the linear system operator is bounded and therefore continuous.

A necessary and sufficient condition for asymptotically stable is that the real parts of all the eigenvalues of matrix $\underline{A}$ are negative. This is a desired property for linear systems.

Before proving a convergence theorem which guarantees that $\tilde{S}$ is a good approximation to $\tilde{S}_o$ provided that the norm of $\underline{D}$ is small enough, a design proposition is presented as follows:

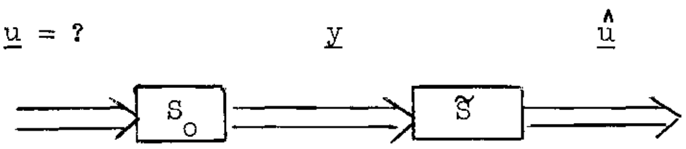
1. Given an I.F.O. linear system $S_o = [\underline{A}, \underline{B}, \underline{C}]$, construct a linear system $S = [\underline{A}, \underline{B}, \underline{C}, \underline{D}]$ where $\underline{D}$ has left inverse.

2. Choose the elements of $\underline{D}$ as small as possible, and construct an inverse system $\tilde{S}$ of S by the algorithm given at the beginning of this section. Post-cascade $\tilde{S}$ with S_o, then $\tilde{S}$ will serve as an approximation to the true inverse $\tilde{S}_o$ which is hard to be realized in practice.

$\tilde{S}$ will be referred to as a dynamical <u>pseudo-inverse system</u> for S_o. The following theorem will show that $\tilde{S}$ is a good approximation.

<u>Theorem 2</u>: (Convergence) Assume $S_o = [\underline{A}, \underline{B}, \underline{C}]$ is I.F.O. and asymptotically stable, and let $\underline{u}(t)$ be the unknown input of S_o, $\underline{y}(t)$ be the corresponding output.

If we post-cascade the pseudo-inverse system $\tilde{S}$ with S_o, and let $\hat{\underline{u}}(t)$ be the output of $\tilde{S}$ (turns out to be an estimation of $\underline{u}(t)$), i.e.



then

$$\|\hat{\underline{u}}(t) - \underline{u}(t)\| \to 0$$

as

$$\|\underline{D}\| \to 0$$

provided that all the inputs are bounded functions on [0,T].

<<u>Proof</u>>: Define the linear operator L as

$$(\underline{L}\ \underline{u})(t) \triangleq \underline{C}\int_0^t \Phi(t-\tau)\underline{B}\ \underline{u}(\tau)d\ \tau$$

$$= \underline{y}(t)$$

then

$$\underline{L}\ \underline{u} = \underline{y}$$

and

$$(\underline{L} + \underline{D})\hat{\underline{u}} = \underline{y}$$

Subtracting the two equalities:

$$L(\underline{u} - \hat{\underline{u}}) = \underline{D}\ \hat{\underline{u}}$$

$$\| L(\underline{u} - \hat{\underline{u}}) \| = \|\underline{D}\ \hat{\underline{u}}\|$$

$$\leq \|\underline{D}\|\ \|\hat{\underline{u}}\| \qquad \text{by def. of } \|\underline{D}\|$$

Note that S_o is I.F.O. implies L^{-1} exists and

$$L^{-1}(\underline{D}\ \hat{\underline{u}}) = \underline{u} - \hat{\underline{u}}$$

Since, by assumption, S_o is asymptotically stable, by Lemma 3 and Lemma 2 of VIII-3 L is continuous.

Then by Lemma 1 and the Banach Inverse Theorem (Theorem 1, VIII-3), L^{-1} is also continuous, i.e.,

$$\|(\underline{u} - \hat{\underline{u}}) - \underline{L}^{-1}(\underline{0})\| \to 0$$

as

$$\|\underline{D}\ \hat{\underline{u}}\| \to 0$$

Note that S_o is I.F.O. implies

$$\underline{L}^{-1}(\underline{0}) = \underline{0}$$

Hence, we have

$$\|\underline{u} - \hat{\underline{u}}\| \to 0$$

as

$$\|\underline{D}\| \to 0 \qquad \text{(which implies } \|\underline{D}\,\hat{\underline{u}}\| \to 0\text{)}$$

The remaining problem is how to estimate the error induced by such an approximation.

The estimation error is

$$\underline{u} - \hat{\underline{u}} = \underline{L}^{-1}(\underline{D}\,\hat{\underline{u}})$$

and the error norm is

$$\begin{aligned} \|\underline{u} - \hat{\underline{u}}\| &= \|L^{-1}\underline{D}\,\hat{\underline{u}}\| \\ &\le \|\underline{L}^{-1}\underline{D}\|\,\|\hat{\underline{u}}\| \\ &\le \|\underline{L}^{-1}\|\,\|\underline{D}\|\,\|\hat{\underline{u}}\| \end{aligned}$$

We can define the "noise to signal" ratio as

$$\begin{aligned} \text{N/S ratio} &\triangleq \frac{\|\underline{u} - \hat{\underline{u}}\|}{\|\underline{u}\|} \\ &\simeq \frac{\|\underline{u} - \hat{\underline{u}}\|}{\|\hat{\underline{u}}\|} \\ &\simeq \|\underline{L}^{-1}\|\,\|\underline{D}\| \end{aligned}$$

so the problem turns out to be how to estimate $\| \underline{L}^{-1} \|$. Although we know $\| L^{-1} \|$ is bounded, in order to control the error or N/S ratio, we have to have some idea of $\| L^{-1} \|$ such that we can choose a suitable $\| \underline{D} \|$.

There have been some works dealing with the reachable zone of a system response, e.g. [25], subject to constrained input. However, in general, it is not so easy to find out the exact reachable zone analytically unless for lower ordered systems, we can display it on an analog computer by a bang-bang optimal control.

Generally speaking, if the original system S_o is very stable (i.e. with short transient time constant), the value of $\| L^{-1} \|$ might be larger, and hence $\| \underline{D} \|$ should be chosen smaller.

CHAPTER IX

CONCLUSIONS AND RECOMMENDATIONS

IX-1. Conclusions

In this work, some basic problems concerning functional reproducibility (F.R.) and input functional observability (I.F.O.) of multi-variable dynamical systems such as, dependencies on homogeneous solutions and on time-intervals, structures of output function space, admissibility of output functions, functional attainability, dual relations, etc., were investigated. Attentions were concentrated on linear time-invariant dynamical systems and problems were dealt with from the frequency domain view point.

The theoretical foundations were based on the formalization of the finite-dimensional rational function vector space over rational function field. Without this special choice of field, infinite-dimensional problem would arise, and all the subsequent results would not hold. The frequency domain criteria and functional attainability of a given system were based on this finite-dimensional vector space formalization. In particular, the structural properties of a major class of interesting input and output functions (i.e. rational functions) were characterized. Then, the concept of admissible input and output functions was introduced such that the set of all admissible output functions was embedded in the rational function subspace. With the aid of a generalized inner product, the orthogonality, decomposition, adjoint operator, etc. could be defined, and therefore a complete dual theorem for F.R. and I.F.O. was established.

On the other hand, the effect of feedback compensation on F.R. was investigated which then contributed to the discussion of the existence of F.R. or I.F.O. systems. The problems such as, for given system matrix $\underline{A}$, the characteristics of the class of all output matricies $\underline{C}$ such that there exist input matrices $\underline{B}$ making the system F.R. etc. were discussed. Besides, a pseudo-inverse system design (without differentiators) which can approximate the true inverse system (with differentiators) to any desired accuracy was presented. It can be seen that this design approach would be rather useful when the true inverse system with differentiators does not work well.

IX-2. Recommendations

In the process of the investigation, a number of interesting extensions of this research became apparent, the following are some subjects and problems suggested for future study:

(i) Consideration of a larger class of input and output functions.

Since a class of scalar functions which is larger than the class of scalar rational functions would no longer be a field, the algebraic structure composed of the class of all vector-valued functions over the class of scalar functions would no longer be a vector space either. However, the former class satisfies the requirements of an additive Abelian group, and the latter class satisfies the requirements of a ring, and therefore the combination would be a module [26]. Consequently, it is suggested to explore the application of the theory of modules in this context.

(ii) The characterization of all input matrices $\underline{B}$ such that the system $[\underline{A},\underline{B},\underline{C}]$ with given $\underline{A}$ and $\underline{C}$ is F.R. is suggested to be investigated. Then, the F.R. and I.F.O. can be extended to the probabilistic parameter systems.

(iii) Some experiments comparing the pseudo-inverse system and the true inverse system are suggested.

(iv) It is suggested to apply the inverse system to various physical systems measurement. It is also believed that many applications can be found in biomedical and engineering systems diagnostic analysis.

BIBLIOGRAPHY

1. R. E. Kalman, P. L. Falb and M. A. Arbib, Topics in Mathematical Systems Theory, McGraw Hill Co., N. Y., 1969.

2. R. W. Brockett and M. D. Mesarovic, "The Reproducibility of Multivariable Systems," Joint Automatic Control Conference, 1964, pp. 481-486.

3. H. H. Rosenbrock, State Space and Multivariable Theory, Nelson, London, 1970.

4. J. L. Massey and M. K. Sain, "Codes, Automata and Continuous Systems: Explicit Interconnections," IEEE Trans. on Automatic Control, Vol. AC-12, No. 6, pp. 644-650, December 1967.

5. J. L. Massey and M. K. Sain, "Inverse Problem in Coding Automata and Continuous Systems," Proc. of the 8th Ann. Symposium on Switching and Automata Theory, Austin, Texas, pp. 226-232, October 1967.

6. S. P. Singh and R. Liu, "A Survey of Inverse Systems," Proc. Nat. Electron. Conf. 1970, pp. 380-384.

7. P. L. Falb and W. A. Wolovich, "Decoupling in the Design and Synthesis of Multivariable Control Systems," IEEE Trans. on Automatic Control, Vol. AC-12, No. 6, pp. 651-659, December 1967.

8. B. D. Anderson and J. B. Moore, "State Estimation Via the Whitening Filter," Joint Automatic Control Conf., 1968, pp. 123-129.

9. M. K. Sain, "Functional Reprodicibility and the Existence of Classical Sensitivity Matrices," IEEE Trans. on Automatic Control (correspondence), Vol. AC-12, p. 458, August 1967.

10. J. B. Cruz and W. R. Perkins, "A New Approach to the Sensitivity Problem in Multivariable Feedback System Design," IEEE Trans. Automatic Control, Vol. AC-9, pp. 216-223, July 1964.

11. L. G. Birta and I. H. Mufti, "Some Results on an Inverse Problem in Multivariable Systems," IEEE Trans. on Automatic Control, Vol. AC-12, pp. 99-101, February 1967.

12. H. W. Bode and C. E. Shannon, "A Simplified Derivation of Linear Least Square Smoothing and Prediction Theory," Proceedings of the IRE, Vol. 38, No. 4, April 1950, pp. 417-425.

13. R. W. Brockett, "Poles, Zero and Feedback: State Space Interpretation," IEEE Trans. on Automatic Control, Vol. AC-10, No. 2, pp. 129-135, April 1965.

14. D. C. Youla and P. Dorato, "On the Inverse of Linear Dynamical Systems," Polytechnic Institute of Brooklyn, New York, Electrophysics Memorandum, PIBMRI-1319-66, March 1966. Also, IEEE Trans. on Systems Science and Cybernatics, Vol. SSC-5, No. 1, pp. 43-48, January 1969.

15. L. M. Silverman, "Properties and Applications of Inverse Systems," IEEE Trans. on Automatic Control, Vol. AC-13, pp. 436-437, August 1968.

16. L. M. Silverman, "Inversion of Multivariable Linear Systems," IEEE Trans. on Automatic Control, Vol. AC-14, No. 3, pp. 270-276, June 1969.

17. M. K. Sain and J. L. Massey, "Inverses of Linear Sequential Circuits," IEEE Trans. on Computers, Vol. C-17, No. 4, pp. 330-337, April 1968.

18. M. K. Sain and J. L. Massey, "Invertibility of Linear Time-invariant Dynamical Systems," IEEE Trans. on Automatic Control, Vol. AC-14, No. 2, pp. 141-149, April 1969.

19. C. A. Desoer, Notes for a Second Course on Linear Systems, Van Nostrand Reinhold Company, N.Y., 1970.

20. A. Ralson, A First Course in Numerical Analysis, McGraw-Hill Company, N.Y., 1965.

21. C. R. Rao and S. K. Mitra, Generalized Inverse of Matrices and Its Applications, John Wiley and Sons, N.Y., 1971.

22. D. Luenberger, Optimization by Vector Space Methods, John Wiley and Sons, N.Y., 1969.

23. A. S. Jackson, Analog Computation, McGraw-Hill Company, N.Y., 1960.

24. J. L. Willems, Stability Theory of Dynamical Systems, Nelson, London, 1970.

25. J. L. LeMay, "Recoverable and Reachable Zones for Control Systems with Linear Plants and Bounded Controller Outputs," IEEE

Trans. on Automatic Control, Vol. AC-9, No. 4, pp. 346-354, October 1964.

26. S. McLane and G. Birkhoff, Algebra, MacMillan, N.Y., 1967.